Lecture 28

11.8 - Power Series

Def: A power series is a series of the form

$$c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

where the c_j's are real numbers.

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{4^n}$. This series converges for some x and diverges for others. This is a geometric series with $r = \frac{x}{4}$, so as long as $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$, it converges.

A power series defines a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

whose domain is the set of all for which the series converges.

Ex: Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{4^n}$. What is the domain of f? What is $f(0)$?

$D(f) = (-4, 4)$ (from above) $\quad f(0) = 1 + \frac{0}{4} + \frac{0^2}{16} + \cdots = 1$

Def: A <u>power series centered at a</u> is a power series of the form:

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Ex: Around what value is the power series

$$\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$$

centered? For what x does the series converge?

$$\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3} = \sum_{n=1}^{\infty} \frac{5^n(x-\frac{4}{5})^n}{n^3}$$ Centered at $a = \frac{4}{5}$.

Use ratio test to test convergence:

needed for convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}(x-\frac{4}{5})^{n+1}}{(n+1)^3}\cdot\frac{n^3}{5^n(x-\frac{4}{5})^n}\right| = \lim_{n\to\infty}\left|\frac{5(x-\frac{4}{5})n^3}{(n+1)^3}\right| = 5\left|x-\frac{4}{5}\right| < 1$$

$$\Rightarrow -1 < 5x-4 < 1 \Rightarrow 3 < 5x < 5 \Rightarrow \frac{3}{5} < x < 1$$

When $x = \frac{3}{5}$ or 1, the limit above is 1, so we check those separately:

$\boxed{x=\frac{3}{5}}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$: converges by alt. series test. ✓

$\boxed{x=1}$ $\sum_{n=1}^{\infty} \frac{1}{n^3}$: converges by p-series test.

So, the power series converges for $\frac{3}{5} \le x \le 1$.

Theorem: For any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are 3 possibilities for the values of x for which the series converges:

1) The series converges only for $x=a$
2) The series converges for all x
3) There is a number $R>0$ such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. The case(s) when $|x-a|=R$ must be checked individually.

Def: The Radius of Convergence of a power series is, as broken down in the 3 cases above:

1) 0 2) ∞ 3) R.

The Interval of Convergence is the interval of x values for which the series converges. In the 3 cases

1) $\{x\}$ 2) $(-\infty, \infty)$

3) can be any of:

$(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, $[a-R, a+R]$

Tip: It's usually best to use the ratio test to try to find the radius of convergence (and hence interval of convergence).

Ex: Find the radius of convergence and interval of convergence for the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ (c) $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

(a) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n+2)}\right| = 0 < 1$$

So, the series converges for all x, i.e.,

R.O.C. $= \infty$, I.O.C. $= (-\infty, \infty)$

(b) $$\lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\left|\frac{xn}{n+1}\right| = |x| < 1$$

Need to check endpoints: $x = -1, 1$

$\boxed{x=1}$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$: conv. by alt. series test. ✓

$\boxed{x=-1}$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty}\frac{-1}{n}$ ✗ div. by p-series test.

R.O.C. $= 1$, I.O.C. $= (-1, 1]$.

(c) $$\lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}\left|(n+1)(2x-1)\right| = \infty,$$ unless $x = \frac{1}{2}$ (then $\lim = 0$)

R.O.C. $= 0$, I.O.C. $= \left\{\frac{1}{2}\right\}$